Kramers-Kronig relations

In practice it is not necessary to have a complete, independent knowledge of $\varepsilon'(\omega)$ and $\varepsilon''(\omega)$ as a function of frequency. If one is known for all ω the other can be obtained via the following expressions (assuming electrical ω regime):

$$\varepsilon'(\omega) = \varepsilon_\infty + \frac{2}{\pi}\int_0^\infty \frac{\omega'\varepsilon''(\omega')}{(\omega'^2 - \omega^2)} d\omega'$$

$$\varepsilon''(\omega) = -\frac{2}{\pi}\int_0^\infty \frac{\omega\left[\varepsilon'(\omega') - \varepsilon_\infty\right]}{(\omega'^2 - \omega^2)} \, d\omega'$$

These are <u>the Kramers-Kronig relations.</u>

To demonstrate that this works in the case of the Debye equations consider the application of the expression for $\varepsilon'(\omega)$.

We have that:

$$\varepsilon'(\omega) = \varepsilon_\infty + \frac{2}{\pi}\int_0^\infty \frac{\omega'(\varepsilon_s - \varepsilon_\infty)\omega'\tau}{(1+\omega'^2\tau^2)(\omega'^2 - \omega^2)} \, d\omega'$$

$$= \varepsilon_\infty + \frac{2(\varepsilon_s - \varepsilon_\infty)\tau}{\pi}\int_0^\infty \frac{\omega'^2}{(1+\omega'^2\tau^2)(\omega'^2 - \omega^2)} d\omega'$$

Let $\omega'^2 = x$, $\Rightarrow 2\omega' \, d\omega' = dx$, $d\omega' = dx/(2x^{1/2})$

$$\therefore \varepsilon'(\omega) = \varepsilon_\infty + \frac{2(\varepsilon_s - \varepsilon_\infty)\tau}{\pi} \cdot \frac{1}{2\tau^2} \cdot \int_0^\infty \frac{x^{1/2}}{(1/\tau^2 + x)(x - \omega^2)} dx$$

[See Tables of integrals, series and products, Gradshteyn & Ryzhik.]

$$= \varepsilon_\infty + \frac{2(\varepsilon_s - \varepsilon_\infty)\tau}{\pi} \cdot \frac{1}{2\tau^2} \cdot \left[\frac{\pi}{\tau} \cdot \frac{1}{(1/\tau^2 + \omega^2)}\right]$$

Therefore, as required

$$\varepsilon'(\omega) = \varepsilon_\infty + \frac{(\varepsilon_s - \varepsilon_\infty)}{(1+\omega^2\tau^2)}$$