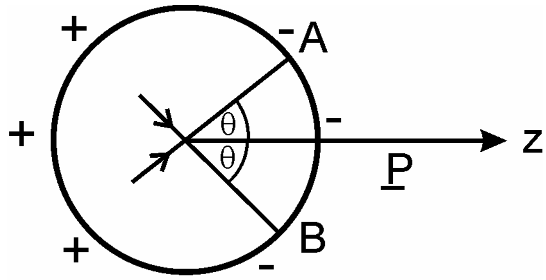
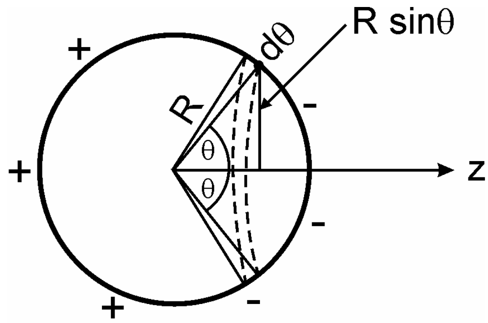
E-field at the centre of a spherical cavity within a block of uniformly polarised dielectric material

The diagram below shows a cross-section through a hollow sphere in a piece of uniformly polarised dielectric material.



By considering elements of charge at A and B, say, where the surface charge density is the same (because of the symmetry of the problem) it is clear that there is only a z direction contribution to the E-field at the centre of the cavity - all of the charge on the surface of the cavity can be paired of in this way so there can be no non z direction contribution.



The surface polarisation charge density in this case is given by $\sigma_{bound} = \underline{P} \cdot \hat{\underline{n}} = -P\cos\theta$ (the - sign is due to the fact that $\hat{\underline{n}}$ is an inward pointing unit vector).

The area of the annulus between θ and $\theta + d\theta$ is $Rd\theta.2\pi R\sin\theta$ which $\Rightarrow$ charge on surface of this annulus is $dq = -P\cos\theta.2\pi R^2\sin\theta d\theta$. The contribution to the z component of the E-field is thus given by

$$dE_z = \frac{-dq}{4\pi\varepsilon_o R^2} \cdot \cos\theta$$

$$= \frac{P\sin\theta\cos^2\theta d\theta}{2\varepsilon_o}$$

The total field at the centre of the cavity in the z direction thus has the value

$$E_z = \frac{P}{2\varepsilon_o}\int_0^{\pi}\sin\theta\cos^2\theta d\theta = \frac{P}{2\varepsilon_o}\left[\frac{-\cos^3\theta}{3}\right]_0^{\pi} = \frac{P}{3\varepsilon_o}$$