ELECTROMAGNETISM : Course synopsis

Introduction

Brief review of vector knowledge required. Divergence, gradient, curl. Divergence theorem and Stokes theorem + vector identities needed: $\nabla \times \nabla \times \underline{A} = \nabla(\nabla.\underline{A}) - \nabla^2\underline{A}$, $\nabla.(\underline{A} \times \underline{B}) = \underline{B}.(\nabla \times \underline{A}) - \underline{A}.(\nabla \times \underline{B})$, $\nabla.(\nabla \times \underline{A}) = 0$ always, $\nabla \times (\nabla A) = 0$ always.

Phasor representation. Time-averaged power and related quantities in terms of phasor representation, e.g. $\langle P \rangle = \frac{1}{2}$ Real $[V(t) . I^*(t)]$.

Reminder of standard expressions for electric and magnetic (Biot-Savart) fields and forces. Lorentz force expression $\underline{F} = q(\underline{E} + \underline{v} \times \underline{B})$

Formulation of differential equation form of Maxwell's Equations:

$$1)\ \nabla.\underline{E} = \frac{\rho}{\varepsilon_o} \qquad \textit{from Gauss' Law}$$

$$2)\ \nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \qquad \textit{from Faraday's Law of Induction}$$

$$3)\ \nabla.\underline{B} = 0 \qquad \textit{no magnetic monopoles}$$

$$4)\ \nabla \times \underline{B} = \mu_o \underline{J} + \mu_o \varepsilon_o \frac{\partial \underline{E}}{\partial t} \qquad \textit{beginning with Ampere's circuital Law}$$

Reduction to two separate pairs describing electrostatics and magnetostatics when there is no time dependence. Brief references to Laplace's eqn., Poissons eqn. Mention of vector potential defined via $\underline{B} = \nabla \times \underline{A}$.

The Continuity Equation:

$$\nabla.\underline{J} = -\frac{\partial \rho}{\partial t}$$

and charge conservation.

Propagation of EM waves in a vacuum

3-D generalisation of wave eqn. as appropriate for EM waves in free space with $c = 1/(\mu_0\varepsilon_0)^{1/2}$:

$$\nabla^2 \underline{E} = \mu_o \varepsilon_o \frac{\partial^2 \underline{E}}{\partial t^2}, \qquad \nabla^2 \underline{B} = \mu_o \varepsilon_o \frac{\partial^2 \underline{B}}{\partial t^2}$$

Plane-wave solutions with $\underline{E}, \underline{B} \propto e^{i(\omega t - \underline{k}.\underline{r})}$, $k^2 = \omega^2\mu_0\varepsilon_0$. Phase and group velocities. Mutual orthogonality of $\underline{E}$, $\underline{B}$ and $\underline{k}$ with $\underline{E} \times \underline{B}$ in direction of propagation ($\underline{k}$) and $B = E/c$.

Media

Dielectric materials. Dipole moment $\underline{p}$, and polarisation $\underline{P}$. Derivation of expressions for surface charge density $\sigma_b = \underline{P}.\hat{\underline{n}}$, volume charge density $\rho_b = -\nabla.\underline{P}$ and polarisation current density $\underline{J}_b = \partial\underline{P}/\partial t$. The electric displacement, $\underline{D}$. Electric susceptibility, χ_e, with $\underline{P} = \varepsilon_0\chi_e\underline{E}$ in linear, isotropic, homogeneous (LIH) media.

Magnetic materials. Magnetic dipole moment $\underline{m}$ and magnetisation $\underline{M}$. Derivation of expressions for volume current density $\underline{J}_M = \nabla \times \underline{M}$ and surface current density $\underline{\lambda}_M = \underline{M} \times \hat{\underline{n}}$. The magnetic field intensity, $\underline{H}$. Magnetic susceptibility, χ_m, with $\underline{M} = \chi_m\underline{H}$ in LIH media.

Conductivity and decay of excess charge density within an LIH conductor. Relaxation time $\tau = \varepsilon/\sigma$.

Maxwell's equations in media

Reformulation and solutions of Maxwell's eqns. in a general LIH medium:

1) $\nabla.\underline{E} = \dfrac{\rho_{free}}{\varepsilon} \approx 0$

2) $\nabla \times \underline{E} = -\dfrac{\partial \underline{B}}{\partial t}$

3) $\nabla.\underline{B} = 0$

4) $\nabla \times \underline{B} = \mu \underline{J}_{free} + \mu\varepsilon\dfrac{\partial \underline{E}}{\partial t} = \mu\sigma\underline{E} + \mu\varepsilon\dfrac{\partial \underline{E}}{\partial t}$

Generalised form of wave equations:

$$\nabla^2\begin{Bmatrix}\underline{E}\\ \underline{B}\\ \underline{H}\end{Bmatrix} - \mu\varepsilon\frac{\partial^2}{\partial t^2}\begin{Bmatrix}\underline{E}\\ \underline{B}\\ \underline{H}\end{Bmatrix} - \mu\sigma\frac{\partial}{\partial t}\begin{Bmatrix}\underline{E}\\ \underline{B}\\ \underline{H}\end{Bmatrix} = 0$$

Specific solutions of $k^2 = \mu\varepsilon\omega^2 - i\mu\sigma\omega$ in the poor ($\varepsilon\omega \gg \sigma$) and good ($\sigma \gg \varepsilon\omega$) conductor limits. Real (n) and complex ($n^* = n_r - in_i$) refractive indices, in both cases refractive index defined as ck/ω.

For a good conductor $n_r = n_i = c(\mu\sigma/2\omega)^{1/2}$, skin depth $\delta = \left(\frac{2}{\mu\sigma\omega}\right)^{\frac{1}{2}}$, $v_{ph} = \omega/k_r = c/n_r = (2\omega/\mu\sigma)^{1/2} = v_g/2.$ Copper and seawater examples. Transparency of "poor" metals to EM waves in the UV region.

Energy flow and Poynting's vector

Poynting's vector $\underline{S} = \underline{E} \times \underline{H}$. Proof that $P = \int \underline{S}.d\underline{a} = \int (\underline{E} \times \underline{H}).d\underline{a}$ leads to consistent statement of conservation of energy. Current-carrying wire and simple plane-wave examples. Time-averaged Poynting vector, $\underline{S}_{ave} = \frac{1}{2}\mathrm{Re}\left[\underline{E}\times\underline{H}^*\right]$ when phasor representation employed. $S_{ave} = \frac{E_o H_o}{2} = \frac{E_o^2}{2\mu_o c}$ for plane-waves in free space. Impedance of free space $[=\mu_o c=(\mu_0/\varepsilon_0)^{1/2}]$ and LIH media $[=(\mu/\varepsilon)^{1/2}]$. Ratio of electric/magnetic energy densities for non-conducting LIH media (=1) and good conductors ($=\omega\varepsilon/\sigma$).

Reflection/refraction at the interface between two media

Derivation of boundary conditions - continuity of tangential components of $\underline{E}$ and $\underline{H}$ at the interface.

Use of boundary condition on $\underline{E}$ to show constancy of frequency for the incident, reflected and transmitted (plane) waves. Proof that $\theta_I = \theta_R$ and Snell's Law result (independent of polarisation). Total internal reflection (TIR), critical angle and existence of evanescent wave with associated decay length. Frustrated TIR.

Dielectric/metallic (non-magnetic) interface, $\theta_T \approx 0$ independent of θ_I $\Rightarrow$ exponentially decaying wave normal to interface.

Derivation of Fresnel's eqns. with $\underline{E}$ polarised normal and parallel to plane of incidence [makes use of $H = nE/(\mu c)$]. Simplified form in the case of non-magnetic media. Dependence of phase on relative values of refractive indices. Brewster angle. Normal incidence. Reflection (R) and transmission (T) coefficients. $R + T = 1$.

Low pressure ionised gases - plasmas

Examples of plasmas. Response of low pressure ($\Rightarrow$ no collisions/energy transfer) plasma to a plane EM field. Plasma conductivity = $-iNq^2/(\omega m)$. No net absorption of energy once oscillation established due to relative phase of E, J. Use of $k^2 = \omega^2\mu\varepsilon - i\mu\sigma\omega$ to obtain $k^2 = (1-\omega_p^2/\omega^2)\omega^2/c^2$ with $\omega_p = (Nq^2/m\varepsilon_o)^{1/2}$ - the plasma frequency. In hertz, $\nu_p \approx 9N^{1/2}$ Hz. Propagating ($\omega > \omega_p$) and attenuated ($\omega < \omega_p$) solution regimes. Decay length.

Results for the phase and group velocities and refractive index are:

Phase velocity, $v_{\mathrm{ph}} = \dfrac{c}{\left(1 - \dfrac{\omega_{\mathrm{p}}^2}{\omega^2}\right)^{\frac{1}{2}}}$

Group velocity, $v_{\mathrm{g}} = c\left(1 - \dfrac{\omega_{\mathrm{p}}^2}{\omega^2}\right)^{\frac{1}{2}}$

Refractive index, $n = \left(1 - \dfrac{\omega_{\mathrm{p}}^2}{\omega^2}\right)^{\frac{1}{2}}$

(Above applying when $\omega > \omega_p$.)

Ionosphere used as a particular example. Handout giving review of implications for terrestrial (D, E, F, F_1, F_2 layers) and satellite communications.

Radiation theory

Solution of Maxwell's eqns. in time-independent case. Vector potential, $\underline{A}$. Choice of Coulomb gauge in order to obtain expression for $\underline{A}$ by analogy with more familiar expression for scalar potential, V:

$$V(\underline{r}_2) = \frac{1}{4\pi\varepsilon_o}\int \frac{\rho(\underline{r}_1)d\tau_1}{|\underline{r}_2 - \underline{r}_1|}$$

$$\underline{A}(\underline{r}_2) = \frac{\mu_o}{4\pi}\int \frac{\underline{J}(\underline{r}_1)d\tau_1}{|\underline{r}_2 - \underline{r}_1|}$$

Generalisation to time-dependent case - "common sense" approach used rather than deep theoretical study. $\underline{B} = \nabla \times \underline{A}$, $\underline{E} = -\nabla V - \dfrac{\partial \underline{A}}{\partial t}$.

Hertzian/oscillating electric dipole (magnetic dipole not considered). Expression for $\underline{A}$ obtained. Expression for $\underline{B} = \nabla \times \underline{A}$ obtained via derivation on handout. Small r (Biot-Savart-like) and large r (radiation term) limits discussed. $\underline{E}$, $\underline{B}$ and $\underline{S}$ patterns discussed using spherical co-ordinate system expressions. Radiation resistance (R_r), directivity (D) and beam width (W) defined using Hertzian dipole as an example (R_r = 197 Ω, D = 3/2, W = 90°). Realistic antenna/aerial radiating systems, e.g. half-wave dipole (R_r = 73 Ω, D = 1.64, W = 78°) including expressions for $\underline{E}$, $\underline{B}$ - not derived, however, λ/4 monopole + earth screen. Broadside and end-fire arrays briefly discussed and simple reflector + dipole + directors (parasitic element) type Yagi-Uda array as used for tv reception.