Answer to Example Q26

Under steady state conditions ME 4) $\rightarrow \nabla \times \underline{B} = \mu_0 \underline{J}$, $\therefore \underline{J} = \frac{\nabla \times \underline{B}}{\mu_0}$

Substitution in given eqn. leads to

$$\nabla \times \left(\tfrac{1}{\mu_0} \nabla \times \underline{B}\right) = -\frac{nq^2}{M} \underline{B}$$

$$\Rightarrow \nabla \times \nabla \times \underline{B} = -\frac{\mu_0 n q^2}{M} \underline{B}$$

Maths identity

$$\nabla(\nabla \cdot \underline{B}) - \nabla^2 \underline{B} = -\frac{\mu_0 n q^2}{M} \underline{B}$$

ME 3) 0

$$\therefore \nabla^2 \underline{B} = \frac{\mu_0 n q^2}{M} \underline{B} \text{ as required.}$$

Using the (only) z component of suggested solution we have that

$$\nabla^2 B_z(x) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) B_0 e^{-\frac{x}{\lambda}}$$

$$= \frac{B_0}{\lambda^2} e^{-\frac{x}{\lambda}}$$

$$\therefore \nabla^2 B_z(x) = \frac{B_z(x)}{\lambda^2}, \quad \nabla^2 \underline{B} = \frac{1}{\lambda^2} \underline{B}$$

with $\lambda = \left(\frac{M}{\mu_0 n q^2}\right)^{1/2}$

$$\underline{J} = \tfrac{1}{\mu_0} \nabla \times \underline{B} = \tfrac{1}{\mu_0} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ B_0 e^{-\frac{x}{\lambda}} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{B_0 e^{-\frac{x}{\lambda}}}{\mu_0 \lambda} \\ 0 \end{pmatrix} = \underline{J}_0 e^{-\frac{x}{\lambda}}$$

With values given

$$\lambda = \left[\frac{2 \times 9.11 \times 10^{-31}}{4\pi \times 10^{-7} \times 5 \times 10^{28} \times (1.6 \times 10^{-19})^2 \times 2^2}\right]^{1/2} = 1.68 \times 10^{-8}\,m$$

Both $\underline{B}$ field and current density decay exponentially from surface over distance ≈ penetration depth