Answer to Electromagnetism Example Question 23

In the time-dependent case we still have that

$$\nabla \cdot \underline{E} = \frac{\rho}{\varepsilon_o} \qquad \text{[ME 1)]}$$

but now (result obtained in lectures)

$$\underline{E} = -\nabla V - \frac{\partial \underline{A}}{\partial t}$$

Substitution of this in ME 1) leads to

$$-\nabla^2 V - \frac{\partial}{\partial t}\nabla \cdot \underline{A} = \frac{\rho}{\varepsilon_o}$$

If we then use the Lorentz gauge with $\nabla \cdot \underline{A} = -\mu_o \varepsilon_o \frac{\partial V}{\partial t}$ as given in the question we obtain, after rearrangement, our first required result that

$$\nabla^2 V = -\frac{\rho}{\varepsilon_o} + \mu_o \varepsilon_o \frac{\partial^2 V}{\partial t^2}$$

We also have, as usual, from ME 3) that

$$\nabla \cdot \underline{B} = 0 \;\Rightarrow \underline{B} = \nabla \times \underline{A}$$

Substituting this in ME 4) we obtain

$$\nabla \times \underline{B} = \nabla \times \nabla \times \underline{A} = \nabla\left(\nabla \cdot \underline{A}\right) - \nabla^2 \underline{A} = \mu_o \underline{J} + \mu_o \varepsilon_o \frac{\partial \underline{E}}{\partial t}$$

Substituting our Lorentz gauge expression for $\nabla \cdot \underline{A}$ and and again employing $\underline{E} = -\nabla V - \frac{\partial \underline{A}}{\partial t}$ this leads to

$$\nabla\left(-\mu_o \varepsilon_o \frac{\partial V}{\partial t}\right) - \nabla^2 \underline{A} = \mu_o \underline{J} + \mu_o \varepsilon_o \frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \underline{A}}{\partial t}\right)$$

Our judicious choice of the Lorentz gauge leads to cancellation of the term $-\mu_o \varepsilon_o \frac{\partial}{\partial t}\nabla V$ which appears on both sides of our eqn. leading to the second of our desired results, after rearrangement, that

$$\nabla^2 \underline{A} = -\mu_o \underline{J} + \mu_o \varepsilon_o \frac{\partial^2 \underline{A}}{\partial t^2}$$