Answer to EM Example Q22

With $k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)$ and hence $v_g = \frac{d\omega}{dk} = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}$

as given in lectures we can expand using a Taylor series

$$v_g \approx c\left(1 - \tfrac{1}{2}\cdot\frac{\omega_p^2}{\omega^2}\cdots\right) \approx c\left(1 - \frac{\omega_p^2}{2\omega^2}\right)$$ given that $\omega \gg \omega_p$

The same approach gives $v_{ph} = \dfrac{c}{\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2}} \approx c\left(1 + \dfrac{\omega_p^2}{2\omega^2}\right)$

and hence $v_g + v_{ph} \approx 2c$

Assuming that we can use this expression for V_g we have that the difference in arrival time of the two pulses is

$$\Delta T = T_L - T_H \approx \frac{L}{c}\left[\frac{1}{\left(1 - \frac{F_p^2}{2F_L^2}\right)} - \frac{1}{\left(1 - \frac{F_p^2}{2F_H^2}\right)}\right]$$

transit time of lower frequency signal

transit time of higher frequency

distance

now expressing frequencies in Hz.

Again, using Taylor series expansion

$$\Longrightarrow \Delta T \approx \frac{L}{c}\left[\left(1 + \frac{F_p^2}{2F_L^2}\right) - \left(1 + \frac{F_p^2}{2F_L^2}\right)\right]$$

$$= \frac{LF_p^2}{2c}\left[\frac{1}{F_L^2} - \frac{1}{F_H^2}\right]$$

In this case $\Longrightarrow 1.5 = \dfrac{6\times10^{19}F_p^2}{2\times3\times10^8\times10^{12}}\left[\dfrac{1}{110^2} - \dfrac{1}{115^2}\right]$ $\quad(8.264 - 7.561)\times10^{-5}$

$\Longrightarrow \underline{F_p = 1.46\,kHz}$ (clearly confirms $\omega \gg \omega_p$)

As $F_p = 9N^{1/2}$ $\Longrightarrow N = \underline{\underline{2.6\times10^4/m^3}}$

(standard result, lectures)